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Applications of Bochner formula to minimal submanifold of the sphere

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Abstract

The aim of this note is to study some properties of compact minimal submanifolds M^n of the Euclidean sphere S^N. We will give estimates for the first eigenvalue of the Laplacian of M^n as well as present a new estimate for the norm of the second fundamental form for hypersurfaces. Moreover, we obtain a new characterization of the sphere S^n.

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1. Introduction

Given a closed Riemannian manifold M^n let $\psi : M^n \to S^{n+p}$ be a minimal immersion. We will denote the second fundamental form of $\psi(M)$ by B as well as the squared length of B by S. There is a well-known theorem due to Takahashi [10] which asserts that $\Delta\psi + n\psi = 0$, where Δ stands for the Laplacian in the induced metric by ψ. Therefore, n is an eigenvalue of Δ and it was conjectured by Yau [11] that for such embedded hypersurfaces, n is the first eigenvalue of Δ. This occurs for some class of minimal isoparametric hypersurfaces of S^{n+1}, see [8]. It is important to point out that the first global result in the direction of Yau problem is that one obtained by Choi and Wang [4] which states $\lambda_1 \geq (n/2)$, but their technique works only for codimension one. In a recent paper, Barros and Bessa [1] showed that $\lambda_1 \geq (n/2) + C$, where C is a constant which depends on M^n and ψ.

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On the other hand, Chern conjectured that when S is constant it describes a discrete set. The first contribution on this problem was given by Simons [9] where he showed if S satisfies $0 \leq S \leq n/(2-(1/p))$, then either $S = 0$ or $S = n/(2-(1/p))$. When $S = n/(2-(1/p))$ was shown independently by Chern et al. [3] and Lawson [5] that $S = n$ and M^n is a Clifford torus in S^{n+1}. Later on Leung [6] have shown that $S \geq n - \lambda_1$, where λ_1 stands for the first eigenvalue of the Laplacian of M^n. Hence, there is a link between Chern and Yau problems. Our aim in this note is to improve Leung's result for hypersurfaces by showing that $S \geq ((n-k)(n-1)/n(n-k-1))(n-\lambda_1)$, where k stands for the dimension of the kernel of the second fundamental form. Moreover, we will present a characterization of the sphere as well as some preliminaries result in the direction of Yau type problem for codimension greater than one.

Now we will announce our results in the following theorems.

Theorem 1. *Let $\psi : M^n \to S^{n+p}$ be a minimal immersion of a compact Riemannian manifold into the Euclidean sphere S^{n+p}. If f is a first eigenfunction of the Laplacian of M^n associated to the first eigenvalue λ_1, then*

$$\int_M \sum_{i=1}^{n} |B(\nabla f, e_i)|^2 + \int_M |\nabla f|^2 \geq \int_M |\mathrm{Hess}\, f|^2,$$

where $|\mathrm{Hess}\, f|^2$ stands for the norm of the Hessian of f while $\{e_1, \dots, e_n\}$ stands for an orthonormal tangent frame on M^n. Occurring equality if and only if $\lambda_1 = n$. Furthermore, if $\int_M |\mathrm{Hess}\, f|^2 \geq \int_M \sum_{i=1}^{n} |B(\nabla f, e_i)|^2$, then $\lambda_1 \geq n-1$. In particular, $\lambda_1 \geq n-1$ if $\int_M |\nabla f|^2 \geq (n/\lambda_1) \int_M \sum_{i=1}^{n} |B(\nabla f, e_i)|^2$.

Theorem 2. *Let $\psi : M^n \to S^{n+1}$ be a minimal immersion of a compact Riemannian manifold into the Euclidean sphere S^{n+1}. If λ_1 stands for the first eigenvalue of the Laplacian of M^n and* $\dim \mathrm{Ker}\, B = k$, *then we have*

$$\int_M S|\nabla f|^2 \geq \frac{(n-k)(n-1)}{n(n-k-1)}(n-\lambda_1) \int_M |\nabla f|^2.$$

In particular, if S is constant we have $S \geq ((n-k)(n-1)/n(n-k-1))(n-\lambda_1)$.

Theorem 3. *Let $\psi : M^n \to S^{n+p}$ be a minimal immersion of a compact Riemannian manifold into the Euclidean sphere S^{n+p} and f a first eigenfunction associated to the Laplacian of M. Then we have*: (i) *if ∇f lies on* $\mathrm{Ker}\, B$, *then $\psi(M^n)$ is a totally geodesic sphere S^n*; (ii) *if* $\mathrm{Ric}_{M^n}(\nabla f, \nabla f) \geq (n-1)|\nabla f|^2$, *then $\psi(M^n)$ is also a totally geodesic sphere S^n.*

2. Preliminaries

In order to obtain our results, we establish some preliminaries and notations. Let us denote S^{n+p} the Euclidean sphere of constant sectional curvature one. For a Riemannian closed manifold M^n, let us consider an immersion $\psi : M^n \to S^{n+p}$. Let $\{e_1, \dots, e_{n+p}\}$

be an adapted orthonormal frame to M^n. With the usual ranges of indices we know that the second fundamental form B of M^n is given by

$$B(e_i, e_j) = \sum_{\alpha=1}^{p} h_{ij}^{\alpha} e_{\alpha},$$

where $h_{ij}^{\alpha} = \langle A_\alpha e_i, e_j \rangle$ and A_α is the shape operator in the direction e_α.

The Gauss equation is given by

$$R_{jkl}^{i} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \sum_{\alpha=1}^{p} (h_{ik}^{\alpha} h_{jl}^{\alpha} - h_{il}^{\alpha} h_{jk}^{\alpha}).$$

Then we have

$$R_{jij}^{i} = \delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji} + \sum_{\alpha=1}^{p} (h_{ii}^{\alpha} h_{jj}^{\alpha} - h_{ij}^{\alpha} h_{ji}^{\alpha}). \tag{2.1}$$

Taking in account that ψ is a minimal immersion, Eq. (2.1) yields

$$\mathrm{Ric}(e_i, e_j) = (n-1)\delta_{ij} - \sum_{\alpha=1}^{p} \sum_{k=1}^{n} h_{ik}^{\alpha} h_{jk}^{\alpha}. \tag{2.2}$$

We remember now Bochner formula (see e.g. [2]) which states that for a function $f : M^n \to R$ defined on a Riemannian manifold the following relation holds:

$$\tfrac{1}{2}\Delta|\nabla f|^2 = |\mathrm{Hess}\, f|^2 + \mathrm{Ric}_{M^n}(\nabla f, \nabla f) + \langle \nabla f, \nabla(\Delta f) \rangle, \tag{2.3}$$

where Hess stands for the Hessian form, while Ric stands for the Ricci tensor of M^n and the norm of an operator considered here is the Euclidean which is given by $|A|^2 = \mathrm{tr}(AA^*)$.

If I denotes the identity operator on TM, then we have

$$|\mathrm{Hess}\, f - tfI|^2 = |\mathrm{Hess}\, f|^2 - 2tf\Delta f + nt^2 f^2. \tag{2.4}$$

Therefore, if $\Delta f + \lambda f = 0$, we derive for any $t \in \mathbb{R}$

$$\int_M |\mathrm{Hess}\, f - tfI|^2 = \int_M |\mathrm{Hess}\, f|^2 + \left(2t + \frac{n}{\lambda_1} t^2\right) \int_M |\nabla f|^2. \tag{2.5}$$

In particular, setting $t = -\lambda_1/n$ on Eq. (2.5), we get

$$\int_M |\mathrm{Hess}\, f|^2 = \int_M \left|\mathrm{Hess}\, f + \frac{\lambda_1}{n} fI\right|^2 + \frac{\lambda_1}{n} \int_M |\nabla f|^2. \tag{2.6}$$

Firstly, we will prove a lemma which enables us to derive the proof of Theorems 1 and 3. More precisely we have the following lemma.

Lemma 1. *Let $\psi : M^n \to S^{n+p}$ be a minimal immersion of a compact Riemannian manifold into the Euclidean sphere S^{n+p}. Let f be a first eigenfunction associated to the Laplacian of M^n. If $\{e_1, \ldots, e_n\}$ denotes an orthonormal tangent frame on M^n, then*

$$(n - \lambda_1) \int_M |\nabla f|^2 = \int_M \sum_{i=1}^{n} |B(\nabla f, e_i)|^2 + \int_M |\nabla f|^2 - \int_M |\mathrm{Hess}\, f|^2. \tag{2.7}$$

In particular, we have

$$\int_M \sum_{i=1}^{n} |B(\nabla f, e_i)|^2 - \int_M \left|\text{Hess } f + \frac{\lambda_1}{n} fI\right|^2 = \frac{(n-1)(n-\lambda_1)}{n} \int_M |\nabla f|^2. \tag{2.8}$$

Proof. Eq. (2.2) yields $\text{Ric}(f_i e_i, f_j e_j) = (n-1)\delta_{ij} f_i f_j - \sum_{\alpha=1}^{p} \sum_{k=1}^{n} h_{ik}^{\alpha} h_{jk}^{\alpha} f_i f_j$. From where we obtain

$$\text{Ric}(\nabla f, \nabla f) = (n-1)|\nabla f|^2 - \sum_{i=1}^{n} |B(\nabla f, e_i)|^2. \tag{2.9}$$

We suppose now that $\Delta f = -\lambda_1 f$. Hence integrating Bochner formula with the aid of Stokes theorem and Eq. (2.9), we conclude

$$(n-\lambda_1) \int_M |\nabla f|^2 = \int_M \sum_{i=1}^{n} |B(\nabla f, e_i)|^2 + \int_M |\nabla f|^2 - \int_M |\text{Hess } f|^2.$$

From where we obtain the first part of the lemma. On the other hand using Eq. (2.6) on the last equality, we obtain

$$\int_M \left(\sum_{i=1}^{n} |B(\nabla f, e_i)|^2 - \left|\text{Hess } f + \frac{\lambda_1}{n} fI\right|^2 \right) = \frac{(n-1)(n-\lambda_1)}{n} \int_M |\nabla f|^2,$$

which finishes the proof of lemma.

Secondly, we prove a lemma which enables us to derive Theorem 2. □

Lemma 2. *Let $A : V \to V$ be a traceless non-null symmetric linear operator defined over a finite dimensional vector space V. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis which diagonalize A, i.e. $Ae_i = \lambda_i e_i$. If $\dim \operatorname{Ker} A = k$, then for any j we have*

$$\lambda_j^2 \leq \frac{(n-k-1)|A|^2}{(n-k)}.$$

Proof. We may assume without loss of generality that $\lambda_1 = \cdots = \lambda_k = 0$. Hence for $\lambda_j \neq 0$, we have

$$\lambda_j^2 = \left(\sum_{\substack{i=k+1 \\ i \neq j}}^{n} \lambda_i \right)^2 \leq (n-k-1) \sum_{\substack{i=k+1 \\ i \neq j}}^{n} \lambda_i^2.$$

From where we get $(n-k)\lambda_j^2 \leq (n-k-1) \sum_{i=k+1}^{n} \lambda_i^2 = (n-k-1)|A|^2$. Hence we finish the proof of the lemma. □

3. Proof of theorems

Proof of Theorem 1. For the proof of Theorem 1 we proceed as follows. Since Takahashi theorem implies $n \geq \lambda_1$, we have that the left-hand side of Eq. (2.7) of the Lemma 1 is non-negative, so it is also the right-hand side. Therefore we have

$$\int_M \sum_{i=1}^{n} |B(\nabla f, e_i)|^2 + \int_M |\nabla f|^2 \geq \int_M |\text{Hess } f|^2.$$

Furthermore, equality occurs if and only if $\lambda_1 = n$. Moreover, we can write the first equation of Lemma 1 in the following way:

$$\int_M |\text{Hess } f|^2 - \int_M \sum_{i=1}^{n} |B(\nabla f, e_i)|^2 = (\lambda_1 - n + 1) \int_M |\nabla f|^2.$$

Hence $\int_M |\text{Hess } f|^2 \geq \int_M \sum_{i=1}^{n} |B(\nabla f, e_i)|^2$ yields $(\lambda_1 - n + 1) \geq 0$. Finally we note that $\int_M |\nabla f|^2 \geq (n/\lambda_1) \int_M \sum_{i=1}^{n} |B(\nabla f, e_i)|^2$ implies $\int_M |\text{Hess } f|^2 \geq \int_M \sum_{i=1}^{n} |B(\nabla f, e_i)|^2$. From where we complete the proof of Theorem 1. □

Proof of Theorem 2. Let $\{e_1, \dots, e_n\}$ be an orthonormal referential which diagonalize the second fundamental form A, i.e. $Ae_i = k_i e_i$ and let θ_i be the angle between ∇f and e_i. Then we have

$$|B(\nabla f, e_i)|^2 = \langle A\nabla f, e_i\rangle^2 = \langle \nabla f, Ae_i\rangle^2 = k_i^2 \cos^2\theta_i |\nabla f|^2.$$

Using now Eq. (2.7) of Lemma 1, we obtain

$$\int_M \left(\sum_{i=1}^{n} k_i^2 \cos^2\theta_i\right) |\nabla f|^2 = \int_M |\text{Hess } f|^2 + (n - \lambda_1 - 1) \int_M |\nabla f|^2.$$

We apply now Lemma 2 to the last equation to obtain

$$\frac{(n-k-1)}{(n-k)} \int_M S|\nabla f|^2 \geq \int_M |\text{Hess } f|^2 + \int_M (n - 1 - \lambda_1)|\nabla f|^2.$$

Since $\int_M |\text{Hess } f|^2 \geq (\lambda_1/n) \int_M |\nabla f|^2$, we conclude

$$\int_M S|\nabla f|^2 \geq \frac{(n-k)(n-1)}{(n-k-1)n}(n - \lambda_1) \int_M |\nabla f|^2,$$

which finishes the proof of Theorem 2. □

Proof of Theorem 3. At first, we remember a theorem due to Obata [7] which states that a Riemannian manifold M^n is isometric to a unit sphere S^n if and only if it admits a differentiable function f such that Hess $f = -f$, where Hess f stands for the Hessian form.

We suppose now that $\nabla f \in \operatorname{Ker} B$, i.e. $B(\nabla f, e_i) = 0$ for any e_i. Using Eq. (2.8), we have

$$\int_M \left|\operatorname{Hess} f + \frac{\lambda_1}{n} f\right|^2 = \frac{(n-1)(\lambda_1 - n)}{n} \int_M |\nabla f|^2.$$

Since the right-hand side of this last equation is non-positive, we conclude that $\lambda_1 = n$ and therefore $\operatorname{Hess} f = -f$. Now using Obata theorem, we conclude that $\psi(M^n)$ is isometric to a unit sphere S^n and we finish the proof of the first part of Theorem 3.

On the other hand if $\operatorname{Ric}_{M^n}(\nabla f, \nabla f) \geq (n-1)|\nabla f|^2$ according to Eq. (2.9), we derive

$$\int_M (n-1)|\nabla f|^2 - \sum_{i=1}^{n} |B(\nabla f, e_i)|^2 \geq (n-1) \int_M |\nabla f|^2.$$

From where we conclude that $\nabla f \in \operatorname{Ker} B$ and complete the proof of Theorem 3. □

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